

GROUP REPRESENTATION ON REFLEXIVE SPACES

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ABSTRACT. Algebras which admit representations on reflexive Banach spaces seem to be a good generalisation of Arens regular Banach algebras, and behave in a similar way to C^* -algebras and Von Neumann algebras. Such algebras are called weakly almost periodic Banach algebras (or in abbreviated form \mathcal{WAP} -algebras). In this paper, for weighted group convolution measure algebra we construct a representation on reflexive space.

Keywords: WAP-algebra, dual Banach algebra, Arens regularity, weak almost periodicity

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1. INTRODUCTION AND PRELIMINARIES

The dual A^* of a Banach algebra A can be turned into a Banach A -module in a natural way, by setting

$$\langle f \cdot a, b \rangle = \langle f, ab \rangle \text{ and } \langle a \cdot f, b \rangle = \langle f, ba \rangle$$

for all $a, b \in A, f \in A^*$. A *dual Banach algebra* is a Banach algebra A such that $A = (A_*)^*$, as a Banach space, for some Banach space A_* , and such that A_* is a closed A -submodule of A^* ; or equivalently, the multiplication on A is separately weak*-continuous. A functional $f \in A^*$ is said to be *weakly almost periodic* if $\{f \cdot a : \|a\| \leq 1\}$ is relatively weakly compact in A^* . We denote by $\mathcal{WAP}(A)$ the set of all weakly almost periodic elements of A^* . It is easy to verify that, $\mathcal{WAP}(A)$ is a (norm) closed subspace of A^* . As pointed out by Pym[14], $\lambda \in A^*$ is weakly almost periodic if and only if $\lim_m \lim_n \langle a_m b_n, \lambda \rangle = \lim_n \lim_m \langle a_m b_n, \lambda \rangle$ whenever (a_m) and (b_n) are sequences in unit ball of A and both repeated limits exist. For more about weakly almost periodic functionals, see [6]. It is known that the multiplication of a Banach algebra A has two natural but, in general, different extensions (called Arens products) to the second dual A^{**} each turning A^{**} into a Banach algebra. When these extensions are equal, A is said to be (Arens) regular. It can be verified that A is Arens regular if and only if $\mathcal{WAP}(A) = A^*$. Further information for the Arens regularity of Banach algebras can be found in [5, 6]. If A and B are Banach algebras, the linear operator $\phi : A \rightarrow B$ is said to be bounded below if $\inf\{\|\phi(a)\| : \|a\| \geq 1\} > 0$. \mathcal{WAP} -algebras, as a generalization of the Arens

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regular algebras, has been introduced and extensively studied in [8]. Indeed, a Banach algebra A for which the natural embedding of A into $\mathcal{WAP}(A)^*$ is bounded below, is called a *WAP-algebra*. It has also known that A is a *WAP-algebra* if and only if it admits an isomorphic representation on a reflexive Banach space. If A is a *WAP-algebra*, then $\mathcal{WAP}(A)$ separates the points of A and so $\mathcal{WAP}(A)$ is ω^* -dense in A^* . It can be readily verified that every dual Banach algebra, and every Arens regular Banach algebra, is a *WAP-algebra* for comparison see [12]. Moreover group algebras are also always *WAP-algebras*, however, they are neither dual Banach algebras, nor Arens regular in the case where the underlying group is not discrete, see [4, Corollary 3.7] and [17, 12].

The paper is organized as follows. In section 2, we construct a representation on reflexive space.

2. GROUP MEASURE ALGEBRAS

Let G be a locally compact group with left Haar measure λ . A Borel measurable function $\omega \geq 1$ on G is called a weight or weight function if $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$. Throughout this section, let $1 \leq p < \infty$ be a real number and q is such that $1/p + 1/q = 1$, then q is called the exponential conjugate of p . The functions $f : G \rightarrow \mathbb{C}$ such that $f\omega \in L^p(G)$ form a linear space which is denoted by $L^p(G, \omega)$. Then $\|f\|_{p, \omega} = \|f\omega\|_p$ defines a norm on $L^p(G, \omega)$. The dual space of $L^1(G, \omega)$ denoted by $L^\infty(G, 1/\omega)$. It consists of all complex-valued measurable functions g on G such that $g/\omega \in L^\infty(G)$. We equipped $L^\infty(G, 1/\omega)$ with the norm $\|g\|_{\infty, \omega} = \|g/\omega\|_\infty$. Then

$$C_0(G, 1/\omega) = \{f : G \rightarrow \mathbb{C} : f/\omega \in C_0(G)\}$$

is a subspace of it. The dual space of $L^p(G, \omega)$ is $L^q(G, 1/\omega)$ consist of all measurable functions g on G such that $g/\omega \in L^q(G)$ by duality

$$\langle f, g \rangle := \int_G f(x)g(x)d\lambda(x)$$

for all $f \in L^p(G, \omega)$ and $g \in L^q(G, 1/\omega)$.

For measurable functions f and g on G , the convolution multiplication

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) d\lambda(y) \quad (x \in G),$$

is defined at each point $x \in G$ for which this makes sense; i.e., the function $y \mapsto f(y) g(y^{-1}x)$ is λ -integrable. Then $f * g$ is said to exist if $(f * g)(x)$ exists for almost all $x \in G$.

Since $\omega \geq 1$ hence $L^p(G, \omega) \subseteq L^p(G)$ and $L^q(G) \subseteq L^q(G, 1/\omega)$.

The map

$$\gamma_p : L^1(G, \omega) \longrightarrow B(L^p(G, \omega))$$

is defined by $\gamma_p(a)(\xi) = a * \xi$ for all $a \in L^1(G, \omega)$ and $\xi \in L^p(G, \omega)$.

Let $(\gamma_p)_* : L^p(G, \omega) \hat{\otimes} L^q(G, 1/\omega) \longrightarrow L^\infty(G, 1/\omega)$ be such that

$$\begin{aligned} \langle (\gamma_p)_*(\xi \otimes \eta), a \rangle &= \langle (\xi \otimes \eta), \gamma_p(a) \rangle = \langle \eta, \gamma_p(a)(\xi) \rangle \\ &= \langle \eta, a * \xi \rangle = \int_G \int_G \eta(t) a(s) \xi(s^{-1}t) d\lambda(t) d\lambda(s) \\ &= \langle \eta * \check{\xi}, a \rangle \end{aligned}$$

such that $\eta * \check{\xi}(s) = \int_G \xi(s^{-1}t) \eta(t) d\lambda(t)$ and $\check{\xi}(s) = \xi(s^{-1})$ for all $s \in G$. Then $((\gamma_p)_*)^* = \gamma_p$.

Lemma 2.1. *Let G be a locally compact group, and let ω be a weight on G . Let $1 \leq p, q < \infty$.*

- (1) *Every compactly supported function in $L^p(G)$ (respectively $L^q(G)$) belongs to $L^p(G, \omega)$ (respectively $L^q(G, 1/\omega)$).*
- (2) *$C_{00}(G)$ is dense in $L^p(G, \omega)$ (respectively $L^q(G, 1/\omega)$).*

Proof. (1) By [11, Lemma 1.3.3], the weight ω is bounded away from both zero and infinity on compact subsets of G . If $f \in L^p(G)$ with compact support $K = \text{supp} f$ then for all $x \in K$, $\omega(x) < b$ for some $b > 0$. Then $\|f\|_{p, \omega} \leq b \|f\|_p$. Hence $f \in L^p(G, \omega)$.

(2) By (1) $C_{00}(G) \subseteq L^p(G, \omega)$. To show that $C_{00}(G)$ is dense in $L^p(G, \omega)$, let $f \in L^p(G, \omega)$ and $\epsilon > 0$ be given. Since $f\omega \in L^p(G)$, there exists $h \in C_{00}(G)$ such that $\|h - f\omega\|_p^p \leq \epsilon$. Let S denote the compact support of h and observe that $\omega(x) \geq \delta$ for some $\delta > 0$ and all $x \in S$. Since ω is bounded on S , $\omega|_S \in L^p(S)$ and hence there exist a continuous function $\eta : S \rightarrow \mathbb{R}$ such that $\eta(x) \geq \delta$ for all $x \in S$ and

$$\int_S |\eta(x) - \omega(x)|^p d\lambda(x) \leq \frac{\epsilon \delta^p}{\|\eta\|_\infty^p}.$$

Now define a function g on G by $g(x) = \frac{h(x)}{\eta(x)}$ for $x \in S$ and $g(x) = 0$ for $x \notin S$. Since $1/\eta(x) \leq 1/\delta$ for all $x \in S$, it is easily verified that g is continuous on G . Thus $g \in C_{00}(G)$ and

$$\|g - f\|_{p, \omega}^p = \int_S \omega(x)^p |g(x) - f(x)|^p d\lambda(x) + \int_{G \setminus S} \omega(x)^p |f(x)|^p d\lambda(x),$$

We estimate the first integral on the right as follows:

$$\begin{aligned}
\int_S \omega(x)^p |g(x) - f(x)|^p d\lambda(x) &\leq \int_S \omega(x)^p \left| \frac{h(x)}{\eta(x)} - \frac{h(x)}{\omega(x)} \right|^p d\lambda(x) \\
&+ \int_S \omega(x)^p \left| \frac{h(x)}{\omega(x)} - f(x) \right|^p d\lambda(x) \\
&= \int_S \frac{h(x)^p}{\eta(x)^p} |\omega(x) - \eta(x)|^p d\lambda(x) \\
&+ \int_S |h(x) - \omega(x)f(x)|^p d\lambda(x) \\
&\leq \frac{\|h\|_\infty^p}{\delta^p} \int_S |\omega(x) - \eta(x)|^p d\lambda(x) \\
&+ \int_S |h(x) - \omega(x)f(x)|^p d\lambda(x) \\
&\leq \epsilon + \int_S |h(x) - \omega(x)f(x)|^p d\lambda(x).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|g - f\|_{p,\omega}^p &\leq \epsilon + \int_S |h(x) - \omega(x)f(x)|^p d\lambda(x) + \int_{G \setminus S} \omega(x)^p |f(x)|^p d\lambda(x) \\
&= \epsilon + \int_G |h(x) - \omega(x)f(x)|^p d\lambda(x) \leq 2\epsilon.
\end{aligned}$$

This shows that $C_{00}(G)$ is dense in $L^p(G, \omega)$.

For q is similar to p , but g must be defined by $g(x) = \eta(x)h(x)$. \square

Remark 2.2. If $\alpha \leq \omega \leq \beta$, then $L^p(G, \omega) = L^p(G)$ and the norms $\|\cdot\|_{p,\omega}$ and $\|\cdot\|_p$ are equivalent and $C_{00}(G)$ is dense in $L^p(G)$, hence dense in $L^p(G, \omega)$.

For the function ξ on G and $x \in G$ left and right translates $L_x \xi$ and $R_x \xi$ defined by $L_x \xi(y) = \xi(x^{-1}y)$ and $R_x \xi(y) = \xi(yx)$ for all $y \in G$.

Lemma 2.3. *Let G be a locally compact group, and let ω be a weight on G , $1 < p < \infty$ and $\xi \in L^p(G, \omega)$.*

- (1) *for all $x \in G$, $L_x \xi \in L^p(G, \omega)$ and $\|L_x \xi\|_{p,\omega} \leq \omega(x) \|\xi\|_{p,\omega}$.*
- (2) *The map $x \rightarrow L_x$ from G into $L^p(G, \omega)$ is continuous.*
- (3) *Let q be the exponential conjugate of p . If ω is continuous on G and $\eta \in L^q(G, 1/\omega)$. Then $\eta * \xi \in C_0(G, 1/\omega)$.*
- (4) *For every relatively compact neighborhood V of e in G , let $u_V \in L^p(G, \omega)$ be such that $u_V \geq 0$ and $\|u_V\|_{p,\omega} = 1$ and $u_V = 0$ almost every where on $G \setminus V$. Then, given $\xi \in L^p(G, \omega)$ and $\epsilon > 0$,*

$$\|u_V * \xi - \xi\|_{p,\omega} < \epsilon$$

for all sufficiently small V .

Proof. (1) follows simply from submultiplicativity of ω :

$$\begin{aligned}
\|L_x \xi\|_{p,\omega}^p &= \int_G |\xi(x^{-1}t)|^p \omega(t)^p d\lambda(t) \\
&= \int_G |\xi(x^{-1}t)|^p \omega(x^{-1}t)^p \frac{\omega(t)^p}{\omega(x^{-1}t)^p} d\lambda(t) \\
&\leq \omega(x)^p \int_G |\xi(x^{-1}t)|^p \omega(x^{-1}t)^p d\lambda(t) \\
&= \omega(x)^p \|\xi\|_{p,\omega}^p
\end{aligned}$$

(2) Let $\epsilon > 0$. Since $C_{00}(G)$ is dense in $L^p(G, \omega)$. There exists $g \in C_{00}(G)$ such that $\|\xi - g\|_{p,\omega} < \epsilon/3$. Let $x \in G$ and choose a compact neighbourhood V of x in G . Let

$$C = \sup\{\omega(s) : s \in V.\text{supp}g\} < \infty$$

Then for $y \in V$,

$$\begin{aligned}
\|L_x g - L_y g\|_{p,\omega}^p &= \int_G |g(x^{-1}t) - g(y^{-1}t)|^p \omega(t)^p d\lambda(t) \\
&\leq C^p \int_{V.\text{supp}g} |g(x^{-1}t) - g(y^{-1}t)|^p d\lambda(t) \\
&= C^p \|L_x g - L_y g\|_p^p
\end{aligned}$$

when $y \rightarrow x$ converges to zero. Since ω is locally bounded :

$$\begin{aligned}
\|L_x \xi - L_y \xi\|_{p,\omega} &\leq \|L_x(\xi - g)\|_{p,\omega} + \|L_x g - L_y g\|_{p,\omega} + \|L_y(\xi - g)\|_{p,\omega} \\
&\leq (\omega(x) + \omega(y)) \|\xi - g\|_{p,\omega} + \|L_x g - L_y g\|_{p,\omega} < \epsilon
\end{aligned}$$

(3) For $x \in G$, by Hölder inequality:

$$\int_G |\xi(x^{-1}y)\eta(y)| d\lambda(y) = \int_G |L_x \xi(y)| \cdot |\eta(y)| d\lambda(y) \leq \|L_x \xi\|_{p,\omega} \|\eta\|_{q,\omega}.$$

So $\eta * \check{\xi}$ is defined everywhere and bounded on G by $\omega(x) \|\xi\|_{p,\omega} \|\eta\|_{q,\omega}$. For $x, y \in G$, Hölder inequality gives

$$|\eta * \check{\xi}(x) - \eta * \check{\xi}(y)| \leq \|L_x \xi - L_y \xi\|_{p,\omega} \|\eta\|_{q,\omega}.$$

The map $t \rightarrow L_t \xi$ from G into $L^p(G, \omega)$ is continuous and therefore we obtain that $\eta * \check{\xi}$ is continuous. To prove $\eta * \check{\xi} \in C_0(G, 1/\omega)$, note first that $\xi, \eta \in C_{00}(G)$ whenever $\eta * \check{\xi} \in C_{00}(G)$.

If $\xi \in L^p(G, \omega), \eta \in L^q(G, 1/\omega)$, then for $1 \leq r < \infty$, since $C_{00}(G)$ is dense in $L^r(G, \omega)$ and $L^r(G, 1/\omega)$, there exist (ξ_n) and (η_n) in $C_{00}(G)$ such that $\|\xi - \xi_n\|_{p,\omega} \rightarrow 0$ and $\|\eta - \eta_n\|_{q,\omega} \rightarrow 0$. Then for all $x \in G$,

$$\begin{aligned}
|\eta * \check{\xi}(x) - \eta_n * \check{\xi}_n(x)|/\omega(x) &\leq |\eta * (\check{\xi} - \check{\xi}_n)(x)|/\omega(x) + |(\eta - \eta_n) * \check{\xi}(x)|/\omega(x) \\
&\leq \|\xi - \xi_n\|_{p,\omega} \|\eta\|_{q,\omega} + \|\xi\|_{p,\omega} \|\eta - \eta_n\|_{q,\omega},
\end{aligned}$$

which tends to 0 as $n \rightarrow \infty$. It follows that $(\eta * \check{\xi})/\omega \in C_0(G)$.

(4) Since $C_{00}(G)$ is dense in $L^p(G, \omega)$, we can choose $g \in C_{00}(G)$ such that $\|\xi - g\|_{p, \omega} < \frac{\epsilon}{3}$. For g it follows that

$$\begin{aligned}
\|u_V * g - g\|_{p, \omega}^p &= \int_G \left| \int_G u_V(xy) g(y^{-1}) d\lambda(y) - g(x) \right|^p \omega(x) d\lambda(x) \\
&= \int_G \left| \int_G u_V(y) L_y g(x) d\lambda(y) - g(x) \right|^p \omega(x) d\lambda(x) \\
&= \int_G \left| \int_G u_V(y) [L_y g(x) - g(x)] d\lambda(y) \right|^p \omega(x) d\lambda(x) \\
&\leq \int_G \left(\int_G u_V(y) |L_y g(x) - g(x)| d\lambda(y) \right)^p \omega(x) d\lambda(x) \\
&\leq \lambda(V \cdot \text{supp} g) \cdot \sup\{\|L_y g - g\|_\infty^p : y \in V\}.
\end{aligned}$$

Now, since the map $y \rightarrow L_y g$ from G into $L^p(G, \omega)$ is continuous, we find a neighbourhood W of e in G such that, for all $y \in W$,

$$\|L_y g - g\|_\infty \leq \frac{\epsilon}{3\lambda(V \cdot \text{supp} g)^{1/p}}$$

Together with the above estimate we get for all $V \subseteq W$,

$$\begin{aligned}
\|u_V * \xi - \xi\|_{p, \omega} &\leq \|u_V * (\xi - g)\|_{p, \omega} + \|u_V * g - g\|_{p, \omega} + \|g - \xi\|_{p, \omega} \\
&\leq (\|u_V\|_{p, \omega} + 1) \|\xi - g\|_{p, \omega} + \frac{\epsilon}{3} < \epsilon
\end{aligned}$$

□

Now

$$\tilde{\Theta} : M_b(G, \omega) \longrightarrow B(L^p(G, \omega))$$

extends λ_p to $M_b(G, \omega)$ by

$$\langle \eta, \tilde{\Theta}(\mu)(\xi) \rangle = \langle \mu, \eta * \check{\xi} \rangle.$$

Then for $\xi, \eta \in C_{00}(G)$, we have

$$\langle \eta, \tilde{\Theta}(\mu)(\xi) \rangle = \langle \eta, \mu * \xi \rangle$$

where $\mu * \xi(t) = \int_G \xi(s^{-1}t) d\mu(s)$.

The next theorem extends the Young's construction [17, Theorem4] for weighted convolution measure algebras.

Theorem 2.4. *Let G be a locally compact group, and let $\omega \geq 1$ be a continuous weight on G . Then weighted convolution measure algebra $M_b(G, \omega)$ is a \mathcal{WAP} -algebra.*

Proof. Let $\{p_n\}$ be some sequence in $(1, \infty)$ such that $p_n \rightarrow 1$. Let E be the direct sum, in an ℓ_2 -sense, of the spaces $L^{p_n}(G, \omega)$. To be exact,

$$E = \{\{\xi_n\} : \xi_n \in L^{p_n}(G, \omega), \|\{\xi_n\}\|_E < \infty\}$$

with $\|\{\xi_n\}\|_E = (\sum_{n=1}^{\infty} \|\xi_n\|_{p_n, \omega}^2)^{\frac{1}{2}}$, then E is reflexive.

The mapping $\Theta : M_b(G, \omega) \longrightarrow B(E)$ is defined by $\Theta(\mu)(\{\xi_n\}) = \{\mu * \xi_n\}$. Consider the adjoint map $\Theta_* : E \hat{\otimes} E^* \longrightarrow M_b(G, \omega)^*$ given by

$$\begin{aligned} \langle \Theta_*(\xi \otimes \eta), \mu \rangle &= \langle \eta, \Theta(\mu)(\xi) \rangle \\ &= \sum_{n=1}^{\infty} \langle \eta_n, \tilde{\Theta}(\mu)(\xi_n) \rangle \\ &= \sum_{n=1}^{\infty} \langle \mu, \eta_n * \check{\xi}_n \rangle \end{aligned}$$

where $\xi = \{\xi_n\} \in E$, $\eta = \{\eta_n\} \in E^*$ and $\mu \in M_b(G, \omega)$.

In particular, Θ_* maps into $C_0(G, 1/\omega)$. So that Θ is ω^* - ω^* continuous.

For $\xi, \eta \in C_{00}(G)$, we have that

$$\lim_{p \rightarrow 1} \|\xi\|_{p, \omega} = \|\xi\|_{1, \omega} \quad , \quad \lim_{q \rightarrow \infty} \|\eta\|_{q, \omega} = \|\eta\|_{\infty, \omega}.$$

By lemma 2.3 for any $\eta \in C_{00}(G)$ and $\varepsilon > 0$ we can find some $u_V \in C_{00}(G)$ with $\|u_V\|_{1, \omega} = 1$ and $\|\eta * \check{u}_V - \eta\|_{\infty, \omega} < \varepsilon$. As $p_n \rightarrow 1$, we can find $n \in \mathbb{N}$ with $\|u_V\|_{p_n, \omega} < 1 + \varepsilon$ and $\|\eta\|_{q_n, \omega} < (1 + \varepsilon)\|\eta\|_{\infty, \omega}$. It follows that

$$\|\eta * \check{u}_V\|_{\infty, \omega} \leq \|u_V\|_{p_n, \omega} \cdot \|\eta\|_{q_n, \omega} < (1 + \varepsilon)^2 \|\eta\|_{\infty, \omega}$$

and that

$$|\langle \eta, \Theta(\mu)(u_V) \rangle| = |\langle \mu, \eta * \check{u}_V \rangle| \geq |\langle \mu, \eta \rangle| - \varepsilon \|\mu\|$$

By taking suitable supremums, it now follows:

$$\begin{aligned} (1 - \varepsilon) \|\mu\| &= \sup\{|\langle \mu, \eta \rangle| : \eta \in C_{00}(G), \|\eta\| \leq 1\} - \varepsilon \|\mu\| \\ &\leq \sup\{|\langle \eta, \Theta(\mu)(u_V) \rangle| : \eta \in C_{00}(G), \|\eta\| \leq 1\} \\ &= \|\Theta(\mu)(u_V)\| \leq \|\Theta\| \|\mu\| \end{aligned}$$

Since ε is arbitrary Θ is isometric. \square

Let G be a locally compact group, and let ω be a Borel-measurable weight function on it. Then the Fourier-Stieltjes algebra $B(G)$ and the weighted measure algebra $M_b(G, \omega)$ are dual Banach algebras (see [12]), also the Fourier algebra $A(G)$ and $L^1(G, \omega)$ are closed ideals in them respectively. Hence all are \mathcal{WAP} -algebras.

The next example shows that Young's construction can not work for semigroups.

Examples 2.1. Let $S = (\mathbb{N}, \min)$. Then $\ell_1(S)$ is \mathcal{WAP} -algebras. But we can't apply Young's construction for this semigroup. Let $f(n) = \frac{1}{n^2}$ and $g(n) = \frac{1}{\sqrt[3]{n}}$ defined for all $n \in \mathbb{N}$.

$$\|f\|_1 = \sum_{n \in \mathbb{N}} \frac{1}{n^2} < \infty, \|g\|_4 = \sum_{n \in \mathbb{N}} \frac{1}{n^{\frac{4}{3}}} < \infty$$

Then $f \in \ell_1(S)$ and $g \in \ell_4(S)$ but $f * g \notin \ell_4(S)$ since

$$f * g(k) = \sum_{n,m=k} f(n)g(m) = \sum_{m=k} \frac{1}{k^2} \frac{1}{\sqrt[3]{m}} + \sum_{n=k} \frac{1}{n^2} \frac{1}{\sqrt[3]{k}} = \infty$$

for all $k \in \mathbb{N}$.

3. RELATION BETWEEN $C_0(S)$ AND \mathcal{WAP} -ALGEBRA

Let $S = \mathbb{N}$. Then for S equipped with min multiplication, the semigroup algebra $\ell_1(S)$ is a WAP-algebra but is not neither Arens regular nor a dual Banach algebra. While, if we replace the min multiplication with max then $\ell_1(S)$ is a dual Banach algebra (so a WAP-algebra) which is not Arens regular. If we change the multiplication of S to the zero multiplication then the resulted semigroup algebra is Arens regular (so a WAP-algebra) which is not a dual Banach algebra. This describes the interrelation between the concepts of being Arens regular algebra, dual Banach algebra and WAP-algebra.

Definition 3.1. Let X, Y be sets and f be a complex-valued function on $X \times Y$.

- (1) We say that f is a cluster on $X \times Y$ if for each pair of sequences $(x_n), (y_m)$ of distinct elements of X, Y , respectively

$$(3.1) \quad \lim_n \lim_m f(x_n, y_m) = \lim_m \lim_n f(x_n, y_m)$$

whenever both sides of (3.1) exist.

- (2) If f is cluster and both sides of 3.1 are zero (respectively positive) in all cases, we say that f is 0-cluster(respectively positive cluster).

In general $\{f\omega : f \in \text{wap}(S)\} \neq \text{wap}(S, 1/\omega)$. By using [2, Lemma1.4] the following is immediate.

Lemma 3.2. Let $\Omega(x, y) = \frac{\omega(xy)}{\omega(x)\omega(y)}$, for $x, y \in S$. Then

- (1) If Ω is cluster, then $\{f\omega : f \in \text{wap}(S)\} \subseteq \text{wap}(S, 1/\omega)$;
- (2) If Ω is positive cluster, then $\text{wap}(S, 1/\omega) = \{f\omega : f \in \text{wap}(S)\}$.

It should be noted that if $M_b(S)$ is Arens regular (resp. dual Banach algebra) then $M_b(S, \omega)$ is so. We don't know that if $M_b(S)$ is WAP-algebra, then $M_b(S, \omega)$ is so. The following Lemma give a partial answer to this question.

Corollary 3.3. Let S be a locally compact topological semigroup with a Borel measurable weight function ω such that Ω is cluster on $S \times S$.

- (1) If $M_b(S)$ is a WAP-algebra, then so is $M_b(S, \omega)$;
- (2) If $\ell_1(S)$ is a WAP-algebra, then so is $\ell_1(S, \omega)$.

Proof. (1) Suppose that $M_b(S)$ is a WAP-algebra so $\text{wap}(S)$ separates the points of S . By lemma 3.2 for every $f \in \text{wap}(S)$, $f\omega \in \text{wap}(S, 1/\omega)$. Thus the evaluation map $\epsilon : S \rightarrow \tilde{X}$ is one to one.

(2) follows from (1). □

Corollary 3.4. For a locally compact semi-topological semigroup S ,

- (1) If $C_0(S) \subseteq \text{wap}(S)$, then the measure algebra $M_b(S)$ is a WAP-algebra.
- (2) If S is discrete and $c_0(S) \subseteq \text{wap}(S)$, then $\ell_1(S)$ is a WAP-algebra.

Proof. (1) By [3, Corollary 4.2.13] the map $\epsilon : S \rightarrow S^{\text{wap}}$ is one to one, thus $M_b(S)$ is a WAP-algebra.

(2) follows from (1). \square

Dales, Lau and Strauss [7, Theorem 4.6, Proposition 8.3] showed that for a semigroup S , $\ell^1(S)$ is a dual Banach algebra with respect to $c_0(S)$ if and only if S is weakly cancellative. If S is left or right weakly cancellative semigroup, then $\ell^1(S)$ is a WAP-algebra. The next example shows that the converse is not true, in general.

Example 3.5. Let $S = (\mathbb{N}, \min)$ then $\text{wap}(S) = c_0(S) \oplus \mathbb{C}$. So $\ell^1(S)$ is a WAP-algebra but S is neither left nor right weakly cancellative. In fact, for $f \in \text{wap}(S)$ and all sequences $\{a_n\}, \{b_m\}$ with distinct element in S , we have $\lim_m f(b_m) = \lim_m \lim_n f(a_n b_m) = \lambda = \lim_n \lim_m f(a_n b_m) = \lim_n f(a_n)$, for some $\lambda \in \mathbb{C}$. This means $f - \lambda \in c_0(S)$ and $\text{wap}(S) \subseteq c_0(S) \oplus \mathbb{C}$. The other inclusion is clear.

If $\{x_n\}$ and $\{y_m\}$ are sequences in S we obtain an infinite matrix $\{x_n y_m\}$ which has $x_n y_m$ as its entry in the m th row and n th column. As in [2], a matrix is said to be of row type C (resp. column type C) if the rows (resp. columns) of the matrix are all constant and distinct. A matrix is of type C if it is constant or of row or column type C .

J.W.Baker and A. Rejali in [2, Theorem 2.7(v)] showed that $\ell^1(S)$ is Arens regular if and only if for each pair of sequences $\{x_n\}, \{y_m\}$ with distinct elements in S there is a submatrix of $\{x_n y_m\}$ of type C .

A matrix $\{x_n y_m\}$ is said to be upper triangular constant if $x_n y_m = s$ if and only if $m \geq n$ and it is lower triangular constant if $x_n y_m = s$ if and only if $m \leq n$. A matrix $\{x_n y_m\}$ is said to be W -type if every submatrix of $\{x_n y_m\}$ is neither upper triangular constant nor lower triangular constant.

Theorem 3.6. Let S be a semigroup. The following statements are equivalent:

- (1) $c_0(S) \subseteq \text{wap}(S)$.
- (2) For each $s \in S$ and each pair $\{x_n\}, \{y_m\}$ of sequences in S ,

$$\{\chi_s(x_n y_m) : n < m\} \cap \{\chi_s(x_n y_m) : n > m\} \neq \emptyset;$$

- (3) For each pair $\{x_n\}, \{y_m\}$ of sequences in S with distinct elements, $\{x_n y_m\}$ is a W -type matrix;
- (4) For every $s \in S$, every infinite set $B \subset S$ contains a finite subset F such that $\cap\{sb^{-1} : b \in F\} \setminus (\cap\{sb^{-1} : b \in B \setminus F\})$ and $\cap\{b^{-1}s : b \in F\} \setminus (\cap\{b^{-1}s : b \in B \setminus F\})$ are finite.

Proof. (1) \Leftrightarrow (2). For all $s \in S$, $\chi_s \in \text{wap}(S)$ if and only if

$$\{\chi_s(x_n y_m) : n < m\} \cap \{\chi_s(x_n y_m) : n > m\} \neq \emptyset.$$

(3) \Rightarrow (1) Let $c_0(S) \not\subseteq \text{wap}(S)$ then there are sequences $\{x_n\}, \{y_m\}$ in S with distinct elements such that for some $s \in S$,

$$1 = \lim_m \lim_n \chi_s(x_n y_m) \neq \lim_n \lim_m \chi_s(x_n y_m) = 0.$$

Since $\lim_n \lim_m \chi_s(x_n y_m) = 0$, for $1 > \varepsilon > 0$ there is a $N \in \mathbb{N}$ such that for all $n \geq N$, $\lim_m \chi_s(x_n y_m) < \varepsilon$. This implies for all $n \geq N$, $\lim_m \chi_s(x_n y_m) = 0$. Then for $n \geq N$, $1 > \varepsilon > 0$ there is a $M_n \in \mathbb{N}$ such that for all $m \geq M_n$ we have $\chi_s(x_n y_m) < \varepsilon$. So if we omit finitely many terms, for all $n \in \mathbb{N}$ there is $M_n \in \mathbb{N}$ such that for all $m \geq M_n$ we have $x_m y_n \neq s$. As a similar argument, for all $m \in \mathbb{N}$ there is $N_m \in \mathbb{N}$ such that for all $n \geq N_m$, $x_m y_n = s$.

Let $a_1 = x_1$, b_1 be the first y_n such that $a_1 y_n = s$. Suppose a_m, b_n have been chosen for $1 \leq m, n < r$, so that $a_n b_m = s$ if and only if $n \geq m$. Pick a_r to be the first x_m not belonging to the finite set $\cup_{1 \leq n \leq r} \{x_m : x_m y_n = s\}$. Then $a_r b_n \neq s$ for $n < r$. Pick b_r to be the first y_n belonging to the cofinite set $\cap_{1 \leq n \leq r} \{y_n : x_m y_n = s\}$. Then $a_n b_m = s$ if and only if $n \geq m$. The sequences $(a_m), (b_n)$ so constructed satisfy $a_m b_n = s$ if and only if $n \geq m$. That is, $\{a_m b_n\}$ is not of W -type and this is a contradiction.

(1) \Rightarrow (3). Let there are sequences $\{x_n\}, \{y_m\}$ in S such that $\{x_n y_m\}$ is not a W -type matrix, (say) $x_n y_m = s$ if and only if $m \leq n$. Then

$$1 = \lim_m \lim_n \chi_s(x_n y_m) \neq \lim_n \lim_m \chi_s(x_n y_m) = 0.$$

So $\chi_s \notin \text{wap}(S)$. Thus $c_0(S) \not\subseteq \text{wap}(S)$.

(4) \Leftrightarrow (1) This is Ruppert criterion for $\chi_s \in \text{wap}(S)$, see [16, Theorem 4]. \square

Example 3.7. (i) Let S be the interval $[\frac{1}{2}, 1]$ with multiplication $x.y = \max\{\frac{1}{2}, xy\}$, where xy is the ordinary multiplication on \mathbb{R} . Then for all $s \in S \setminus \{\frac{1}{2}\}$, $x \in S$, $x^{-1}s$ is finite. But $x^{-1}\frac{1}{2} = [\frac{1}{2}, \frac{1}{2x}]$. Let $B = [\frac{1}{2}, \frac{3}{4}]$. Then for all finite subset F of B ,

$$\bigcap_{x \in F} x^{-1}\frac{1}{2} \setminus \bigcap_{x \in B \setminus F} x^{-1}\frac{1}{2} = [\frac{2}{3}, \frac{1}{2x_F}]$$

where $x_F = \max F$. By [16, Theorem 4] $\chi_{\frac{1}{2}} \notin \text{wap}(S)$. So $c_0(S \setminus \{\frac{1}{2}\}) \oplus \mathbb{C} \not\subseteq \text{wap}(S)$. It can be readily verified that $\epsilon : S \rightarrow S^{\text{wap}}$ is one to one, so $\ell_1(S)$ is a WAP-algebra but $c_0(S) \not\subseteq \text{wap}(S)$. This is a counter example for the converse of Corollary 3.4.

(ii) Take $T = (\mathbb{N} \cup \{0\}, \cdot)$ with 0 as zero of T and the multiplication defined by

$$n.m = \begin{cases} n & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

Then $S = T \times T$ is a semigroup with coordinate wise multiplication. Now let $X = \{(k, 0) : k \in T\}$, $Y = \{(0, k) : k \in T\}$ and $Z = X \cup Y$. We use the Ruppert criterion [16] to show that $\chi_z \notin \text{wap}(S)$, for each $z \in Z$. Let $B = \{(k, n) : k, n \in T\}$, then $(k, n)^{-1}(k, 0) = \{(k, m) : m \neq n\} = B \setminus \{(k, n)\}$. Thus for all finite subsets F of B ,

$$\begin{aligned} (\cap \{(k, n)^{-1}(k, 0) : (k, n) \in F\}) & \setminus (\cap \{(k, n)^{-1}(k, 0) : (k, n) \in B \setminus F\}) \\ &= (\cap \{(k, 0)(k, n)^{-1} : (k, n) \in F\}) \\ & \setminus (\cap \{(k, n)^{-1}(k, 0) : (k, n) \in B \setminus F\}) \\ &= (B \setminus F) \setminus F = B \setminus F \end{aligned}$$

and the last set is infinite. This means $\chi_{(k,0)} \notin \text{wap}(S)$. Similarly $\chi_{(0,k)} \notin \text{wap}(S)$. Let $f = \sum_{n=0}^{\infty} f(0, n)\chi_{(0,n)} + \sum_{m=1}^{\infty} f(m, 0)\chi_{(m,0)}$ be in $\text{wap}(S)$. For arbitrary fixed n and sequence $\{(n, k)\}$ in S , we have $\lim_k f(n, k) = \lim_k \lim_l f(n, l.k) = \lim_l \lim_k f(n, l.k) = f(n, 0)$ implies $f(n, 0) = 0$. Similarly $f(0, n) = 0$ and $f(0, 0) = 0$. Thus $f = 0$. In fact $\text{wap}(S) \subseteq \ell^\infty(\mathbb{N} \times \mathbb{N})$. Since $\text{wap}(S)$ can not separate the points of S so $\ell_1(S)$ is not a WAP-algebra. Let $\omega(n, m) = 2^n 3^m$ for $(n, m) \in S$. Then ω is a weight on S such that $\omega \in \text{wap}(S, 1/\omega)$, so the evaluation map $\epsilon : S \rightarrow \tilde{X}$ is one to one. This means $\ell_1(S, \omega)$ is a WAP-algebra but $\ell_1(S)$ is not a WAP-algebra. This is a counter example for the converse of Corollary 3.3.

- (iii) Let S be the interval $[\frac{1}{2}, 1]$ with multiplication $x.y = \max\{\frac{1}{2}, xy\}$, where xy is the ordinary multiplication on \mathbb{R} . Then for all $s \in S \setminus \{\frac{1}{2}\}$, $x \in S$, $x^{-1}s$ is finite. But $x^{-1}\frac{1}{2} = [\frac{1}{2}, \frac{1}{2x}]$. Let $B = [\frac{1}{2}, \frac{3}{4}]$. Then for all finite subset F of B ,

$$\bigcap_{x \in F} x^{-1}\frac{1}{2} \setminus \bigcap_{x \in B \setminus F} x^{-1}\frac{1}{2} = [\frac{2}{3}, \frac{1}{2x_F}]$$

where $x_F = \max F$. By [16, Theorem 4] $\chi_{\frac{1}{2}} \notin \text{wap}(S)$. So $c_0(S \setminus \{\frac{1}{2}\}) \oplus \mathbb{C} \subsetneq \text{wap}(S)$. It can be readily verified that $\epsilon : S \rightarrow S^{\text{wap}}$ is one to one, so $\ell_1(S)$ is a WAP-algebra but $c_0(S) \not\subseteq \text{wap}(S)$. This is a counter example for the converse of Corollary 3.4.

- (iv) Take $T = (\mathbb{N} \cup \{0\}, \cdot)$ with 0 as zero of T and the multiplication defined by

$$n.m = \begin{cases} n & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

Then $S = T \times T$ is a semigroup with coordinate wise multiplication. Now let $X = \{(k, 0) : k \in T\}$, $Y = \{(0, k) : k \in T\}$ and $Z = X \cup Y$. We use the Ruppert criterion [16] to show that $\chi_z \notin \text{wap}(S)$, for each $z \in Z$. Let $B = \{(k, n) : k, n \in T\}$, then $(k, n)^{-1}(k, 0) = \{(k, m) : m \neq n\} = B \setminus \{(k, n)\}$. Thus for all finite

subsets F of B ,

$$\begin{aligned}
 (\cap\{(k, n)^{-1}(k, 0) : (k, n) \in F\}) & \setminus (\cap\{(k, n)^{-1}(k, 0) : (k, n) \in B \setminus F\}) \\
 &= (\cap\{(k, 0)(k, n)^{-1} : (k, n) \in F\}) \\
 & \setminus (\cap\{(k, n)^{-1}(k, 0) : (k, n) \in B \setminus F\}) \\
 &= (B \setminus F) \setminus F = B \setminus F
 \end{aligned}$$

and the last set is infinite. This means $\chi_{(k,0)} \notin \text{wap}(S)$. Similarly $\chi_{(0,k)} \notin \text{wap}(S)$. Let $f = \sum_{n=0}^{\infty} f(0, n)\chi_{(0,n)} + \sum_{m=1}^{\infty} f(m, 0)\chi_{(m,0)}$ be in $\text{wap}(S)$. For arbitrary fixed n and sequence $\{(n, k)\}$ in S , we have $\lim_k f(n, k) = \lim_k \lim_l f(n, l.k) = \lim_l \lim_k f(n, l.k) = f(n, 0)$ implies $f(n, 0) = 0$. Similarly $f(0, n) = 0$ and $f(0, 0) = 0$. Thus $f = 0$. In fact $\text{wap}(S) \subseteq \ell^\infty(\mathbb{N} \times \mathbb{N})$. Since $\text{wap}(S)$ can not separate the points of S so $\ell_1(S)$ is not a WAP-algebra. Let $\omega(n, m) = 2^n 3^m$ for $(n, m) \in S$. Then ω is a weight on S such that $\omega \in \text{wap}(S, 1/\omega)$, so the evaluation map $\epsilon : S \rightarrow \bar{X}$ is one to one. This means $\ell_1(S, \omega)$ is a WAP-algebra but $\ell_1(S)$ is not a WAP-algebra. This is a counter example for the converse of Corollary 3.3.

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